A2 Pure Paper 2

October 2021

1. (a) $16 + 20d = 24 \Rightarrow \quad d = 2/5$

(b) $S_n = \frac{1}{2} n (a + l)$

$= \frac{1}{2} \times 500 \times (16 + 16 \cancel{+} 499 \times 2/5)$

$= 57900$

2 (a) $f(x) \leq 7$

(b) $f(1.8) = \quad 7 - 2(1.8)^2 = 13/25$

$g(13/25) = \dfrac{3(13/25)}{5(13/25) - 1} = 39/40$

(c) $y = \dfrac{3x}{5x - 1}$

$(5x - 1)y = 3x$

$5xy - y = 3x$

$5xy - 3x = y$

$x(5y - 3) = y$

$x = \dfrac{y}{5y - 3}$

$g^{-1}(x) = \dfrac{x}{5x - 3}$

③ $\log_3(12y+5) - \log_3(1-3y) = 2$

$$\log_3\left(\frac{12y+5}{1-3y}\right) = 2$$

$$\frac{12y+5}{1-3y} = 3^2$$

$$12y+5 = 9(1-3y)$$

$$12y+5 = 9 - 27y$$

$$39y = 4$$

$$y = 4/39$$

④ Small angles $\sin\theta \approx \theta$ $\cos\theta \approx 1 - \frac{\theta^2}{2}$

$$4\sin\frac{\theta}{2} + 3\cos^2\theta \approx 4\frac{\theta}{2} + 3\left(1-\frac{\theta^2}{2}\right)\left(1-\frac{\theta^2}{2}\right)$$

$$2\theta + 3\left(1 - \theta^2 + \frac{\theta^4}{2}\right) = 2\theta + 3 - 3\theta^2 + \frac{3\theta^4}{2}$$

$a=3$ $b=2$ $c=-3$

(5) $y = 5x^4 - 24x^3 + 42x^2 - 32x + 11$

(a)(i) $\frac{dy}{dx} = 20x^3 - 72x^2 + 84x - 32$

(ii) $\frac{d^2y}{dx^2} = 60x^2 - 144x + 84$

if $x = 1$ $\quad \frac{dy}{dx} = 20 - 72 + 84 - 32 = 0$

$\therefore$ stationary point

$\frac{dy}{dx}$ if $x = 0.9$ $\quad 20(0.9)^3 - 72(0.9)^2 + 84(0.9) - 32$

$\frac{dy}{dx} = -7/50$

if $x = 1.1$ $\quad \frac{dy}{dx} = -1/10$

$\frac{d^2y}{dx^2}$ $\quad x = 1$ $\quad 60 - 144 + 84 = 0$

$\therefore$  must be point of inflection

(6) (a) $\frac{\pi - \theta}{2}$

(b) Area $= \frac{r^2 \cancel{\pi} \theta}{2} = \frac{r^2\left(\frac{\pi-\theta}{2}\right)}{2}$

but want 2 of these so $\frac{\pi r^2 - r^2\theta}{2}$

Centre $\frac{(2r)^2\theta}{2} = 2r^2\theta$

In total Area $= 2r^2\theta + \frac{\pi r^2}{2} - \frac{r^2\theta}{2}$

$= \frac{3r^2\theta + \pi r^2}{2}$

(c) Arc $= r\theta$

So $2r \times \theta + 2\left(\left(\frac{\pi - \theta}{2}\right)r\right) + 4r$

$2r\theta + \pi r - r\theta + 4r$

$r\theta + \pi r + 4r$

(7) (a) $y = x^3 - 10x^2 + 27x - 23$

$$\frac{dy}{dx} = 3x^2 - 20x + 27$$

at $x = 5$ $\quad \frac{dy}{dx} = 2 \quad y = -13$

$$m = \frac{y - y_1}{x - x_1} \qquad 2 = \frac{y + 13}{x - 5}$$

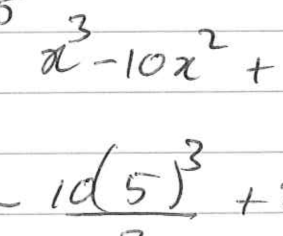
$$2x - 10 = y + 13 \Rightarrow \underline{\underline{y = 2x - 23}}$$

(b)
$$y = 2x - 23$$
$$y = x^3 - 10x^2 + 27x - 23$$

So
$$2x - 23 = x^3 - 10x^2 + 27x - 23$$
$$0 = x^3 - 10x^2 + 25x$$
$$0 = x(x^2 - 10x + 25)$$
$$0 = x(x-5)^2$$

So meets again $x = 0 \quad y = -23$

(c)



$$= \left(\frac{13 + 23}{2}\right) 5 = 90$$

$$\int_0^5 x^3 - 10x^2 + 27x - 23 \, dx = \left[\frac{x^4}{4} - \frac{10x^3}{3} + \frac{27x^2}{2} - 23x\right]_0^5$$

$$\left(\frac{5^4}{4} - \frac{10(5)^3}{3} + \frac{27(5)^2}{2} - 23(5)\right) - 0 = \frac{-455}{12}$$

Final Answer $90 + \frac{-455}{12} = \frac{625}{12}$

(8) $px^3 + qxy + 3y^2 = 26$

differentiate wrt x

$$3px^2 + qy + \frac{dy}{dx}qx + \frac{dy}{dx}6y = 0$$

$$\frac{dy}{dx}(qx+6y) = -3px^2 - qy$$

$$\frac{dy}{dx} = \frac{-3px^2 - qy}{qx+6y}$$

$a = -3 \quad b = -1 \quad c = 6$

$P\ (-1,-4)$ normal $-\left(\frac{qx+6y}{-3px^2-qy}\right)$

so $p(-1)^3 + q(-1)(-4) + 3(-4)^2 = 26$

$-p + 4q + 48 = 26$

$4q - p = -22$

Normal

$$-\left(\frac{-q-24}{-3p+4q}\right) = \frac{q+24}{-3p+4q}$$

Normal $m = \frac{y-y_1}{x-x_1}$

(8) $\dfrac{q+24}{-3p+4q} = \dfrac{y+4}{x+1}$

$(q+24)(x+1) = (y+4)(-3p+4q)$

$\cancel{qx} + \cancel{q} + \cancel{24}x + 24 = -3\cancel{p}y + \cancel{4qy} + 12p + \cancel{16q}$

$(q+24)x + (-4q+3p)y + -15q + 24 + 12p = 0$

$19 \quad x \quad + \quad 26y \quad + \quad 123$

so $q = -5 \qquad p = +2$

9

$$\left(\tfrac{3}{4}\right)^2 \times 1 + \left(\tfrac{3}{4}\right)^3 \times -1 + \left(\tfrac{3}{4}\right)^4 \quad 1$$

$$\left(\tfrac{3}{4}\right)^2 + \left(\tfrac{3}{4}\right)^4 + \left(\tfrac{3}{4}\right)^6 - \left(\left(\tfrac{3}{4}\right)^3 + \left(\tfrac{3}{4}\right)^5 + \left(\tfrac{3}{4}\right)^6 + \cdots\right)$$

$$\frac{\left(\tfrac{3}{4}\right)^2}{1 - \left(\tfrac{3}{4}\right)^2} - \frac{\left(\tfrac{3}{4}\right)^3}{1 - \left(\tfrac{3}{4}\right)^2}$$

$$= \frac{\left(\tfrac{3}{4}\right)^2 - \left(\tfrac{3}{4}\right)^3}{1 - \left(\tfrac{3}{4}\right)^2} = 9/28$$

(10)

$$T = a l^b$$
$$\log T = \log a l^b$$
$$= \log a + \log l^b$$
$$= \log a + b \log l$$

So $\log l$ $\log T$
$(-0.7, 0)$
$(0.21, 0.45)$

$$m = \frac{\Delta y}{\Delta x} = \frac{-0.45}{-0.91} = \frac{45}{91} = b$$
$b = 0.495$

$$m = \frac{y - y_1}{x - x_1} = \frac{y - 0}{x + 0.7}$$

$y = mx + 0.7m$

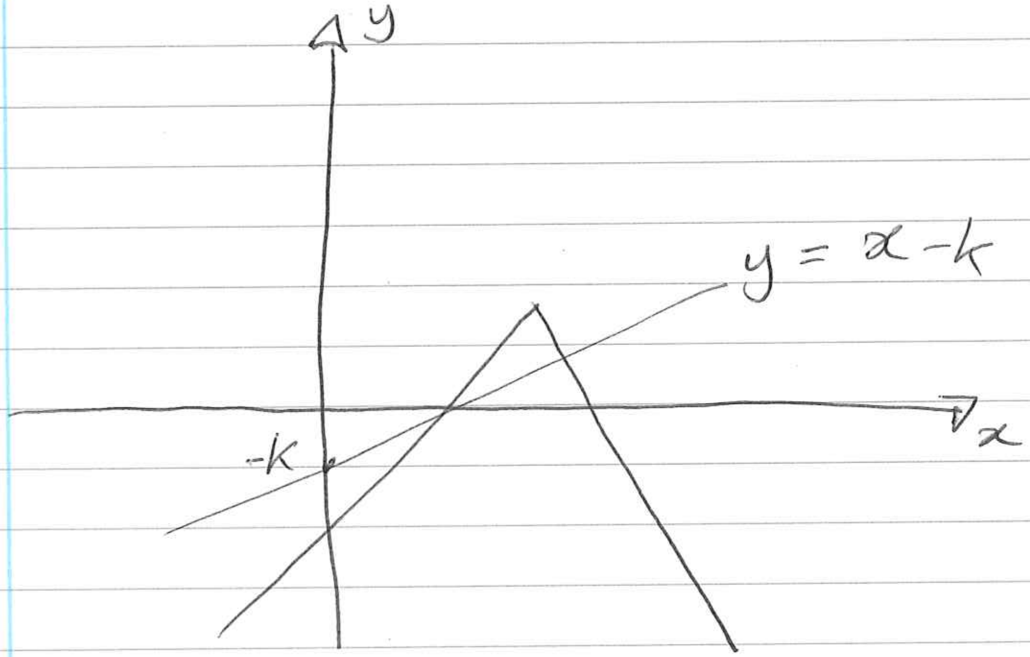
intercept $\frac{9}{26} \approx 0.346$

$\log a = 9/26$ $a = 10^{9/26} \approx 2.22$

So $\boxed{T = 2.22\, l^{0.495}}$

(c) The length of time it would take the pendulum to swing if it is 1m long.

(b) $k - |2x - 3k| > x - k$



Either $\quad x - k = k - 2x + 3k$

$x = 5k/3$

or $\quad x - k = k + 2x - 3k$

$k = x$

So $\quad \frac{5k}{3} > x > k \quad \in \mathbb{R}$

So $\left\{ x : x < \frac{5k}{3} \right\} \cap \left\{ x : x > k \right\}$

(c) $y = 3 - 5(k - |x - 3k|)$

$y = 3 - 5k + 5|x - 3k|$

Min $\quad y = 3 - 5k \qquad x = 3k$

$(3k, 3 - 5k)$

12 $\int_0^{16} \frac{x}{1+\sqrt{x}}\,dx$ let $u = 1 + x^{1/2}$

$\frac{du}{dx} = \frac{1}{2} x^{-1/2}$

$dx = = 2x^{1/2}\,du$

$x = (u-1)^2$

$= \int \frac{(u-1)^2}{u} 2x^{1/2}\,du = \int \frac{(u-1)^2}{u} 2(u-1)\,du$

but if $x = 0$ $u = 1 + 0^{1/2} = 1$

$x = 16$ $u = 1 + 16^{1/2} = 5$

letnew $\int_1^5 \frac{2(u-1)^3}{u}\,du$

$(u-1)^3 = u^3 + 3u^2(-1) + 3u(-1)^2 + (-1)^3$

$\int_1^5 2u^2 - 6u + 6 - 2/u\,du$

$= \left[\frac{2u^3}{3} - 3u^2 + 6u - 2\ln u\right]_1^5$

$= \left(\frac{250}{3} - 75 + 30 - 2\ln 5\right) - \left(2/3 - 3 + 6\right)$

$= \frac{104}{3} - 2\ln 5$

13. $x = \sin 2\theta$ $\quad$ $y = \operatorname{cosec}^3\theta$

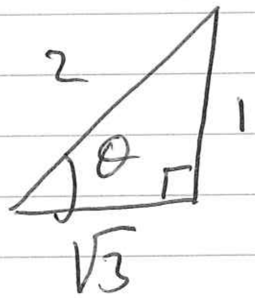
$\frac{dx}{d\theta} = 2\cos 2\theta$ $\quad$ $\frac{dy}{d\theta} = -3\operatorname{cosec}\theta\cot\theta\operatorname{cosec}^2\theta$

$= -3\operatorname{cosec}^3\theta\cot\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{-3\operatorname{cosec}^3\theta\cot\theta}{2\cos 2\theta}$$

if $y = 8 \Rightarrow \operatorname{cosec}\theta = 2$ and $\sin\theta = \frac{1}{2}$

so $\sin\theta = \frac{O}{H}$



$$\frac{dy}{d\theta} = \frac{-3 \times 8 \times \frac{1}{1/\sqrt{3}}}{2\left(\left(\frac{\sqrt{3}}{2}\right)^2 - \frac{1}{4}\right)} = -24\sqrt{3}$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

14 (a) $\frac{dV}{dt} = 0.48 - 0.1h$ m^3/min

$V = 24h$

$\frac{dV}{dh} = 24$

$\frac{dh}{dt} = \frac{dV}{dt} \times 1/\frac{dV}{dh}$

$\frac{dh}{dt} = (0.48 - 0.1h) \times \frac{1}{24}$

$24\frac{dh}{dt} = 0.48 - 0.1h$

$1200\frac{dh}{dt} = 24 - 5h$ $\times 50$

If $t=0$ $h=2$

(b) $\dfrac{dh}{dt} = \dfrac{24-5h}{1200}$

$\dfrac{dt}{dh} = \dfrac{1200}{24-5h}$

$t = -\dfrac{1200}{5} \ln|24-5h| + k$

$\dfrac{5(t-k)}{-1200} = \ln|24-5h|$

$e^{\frac{5(t-k)}{-1200}} = 24-5h$

$h = \dfrac{24}{5} - \dfrac{1}{5} e^{\frac{5(t-k)}{-1200}}$

$h = \dfrac{24}{5} - \dfrac{e^{\frac{5k}{1200}}}{5} e^{\frac{-5t}{1200}}$

$A = \dfrac{24}{5}$ if $t=0$ $h=2$

so $2 = \dfrac{24}{5} - \dfrac{e^{\frac{5k}{1200}}}{5}$

$\Rightarrow \dfrac{e^{5k/1200}}{5} = \dfrac{24}{5} - 2 = \dfrac{14}{5}$

so $h = \dfrac{24}{5} - \dfrac{14}{5} e^{-5t/1200}$ $A = \dfrac{24}{5}$ $B = -14/5$

"k" $= 1/240$

14 (c)

$h = 5$

$$5 = \frac{24}{5} - \frac{14}{5} e^{-t/240}$$

$$14 e^{-t/240} = 24 - 25 = -1$$

not possible as the exponential will never become negative.

(15) (a) $R\cos(\theta+\alpha) = R(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$

$$2\cos\theta \quad - \quad \sin\theta$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\cancel{1}}{\cancel{\cos\theta}} = \frac{1}{2} \Rightarrow \alpha = \tan^{-1}(1/2)$$
$$= \underline{\underline{0.464}}$$

$$R = \sqrt{2^2+1^2} = \underline{\underline{\sqrt{5}}}$$

(b) $H = 3 + 4\cos(0.5t) - 2\sin(0.5t)$

$H = 3 + 2\sqrt{5}\cos(0.5t + 0.464)$

(i) $3 + 2\sqrt{5}$

$$2\pi = 0.5t + 0.464 \Rightarrow \underline{\underline{t = 11.6}}$$

(ii)

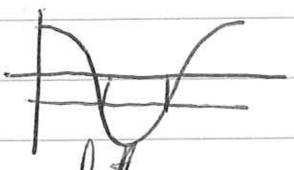
(c) want $H = 0$

$$0 = 3 + 2\sqrt{5}\cos(0.5t + 0.464)$$

let $x = 0.5t + 0.464$

so $0 = 3 + 2\sqrt{5}\cos(x)$

$$x = \cos^{-1}\left(\frac{-3}{2\sqrt{5}}\right) \Rightarrow x = 2.30611078$$



so $x = 2.30611078$

or $2\pi - 2.30611078 = 3.977074528$

So difference is 1.670963748 $\quad \underline{\underline{1.67}}$ 3SF

15

but $x = 0.5t + 0.464$

so $t = 2(x - 0.464)$

so difference is 1.670963748×2

$= 3.341927495$

3.34 to 3 S.F.

d) The $+3$ would need to vary to take account of the height of the water.