1. It is suggested that the sequence $a_k = 2^k + 1, k \ldots 1$ produces only prime numbers.

(a) Show that a_1, a_2 and a_4 produce prime numbers.

(2 marks)

$a_1 = 2^1 + 1 = 3$
$a_2 = 2^2 + 1 = 5$
$a_4 = 2^4 + 1 = 17$
all prime

(b) Prove by counter example that the sequence does not always produce a prime number.

(2 marks)

$a_3 = 2^3 + 1 = 9$

9 isn't prime. since $\frac{9}{3} = 3$

2. Find the angle that the vector $\mathbf{a} = 4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ makes with the positive y-axis.

(3 marks)

$\cos\theta = \frac{y}{|a|}$

$|a| = \sqrt{4^2 + (-1)^2 + 3^2}$
$= \sqrt{26}$

$\cos\theta = \frac{-1}{\sqrt{26}}$

$\theta = 101.3^\circ$

3. $$g(x) = 3\sin\left(\frac{x}{6}\right)^3 - \frac{1}{10}x - 1,\ -40 < x < 20,\ x \text{ is in radians.}$$

(a) Show that the equation $g(x) = 0$ can be written as $x = 6\left(\sqrt[3]{\arcsin\left(\frac{1}{3} + \frac{1}{30}x\right)}\right)$

(3 marks)

$$3\sin\left(\frac{x}{6}\right)^3 = \frac{1}{10}x + 1$$

$$\sin\left(\frac{x}{6}\right)^3 = \frac{1}{30}x + \frac{1}{3}$$

$$\left(\frac{x}{6}\right)^3 = \arcsin\left(\frac{1}{30}x + \frac{1}{3}\right)$$

$$\frac{x}{6} = \sqrt[3]{\arcsin\left(\frac{1}{30}x + \frac{1}{3}\right)}$$

$$x = 6\left(\sqrt[3]{\arcsin\left(\frac{1}{30}x + \frac{1}{3}\right)}\right)$$

(b) Using the formula $x_{n+1} = 6\left(\sqrt[3]{\arcsin\left(\frac{1}{3} + \frac{1}{30}x_n\right)}\right)$, $x_0 = 4$, find, to 3 decimal places, the values of x_1, x_2 and x_3.

(2 marks)

$x_0 = 4$

$x_1 = 4.716$

$x_2 = 4.802$

$x_3 = 4.812$

4. The first 3 terms of a geometric sequence are $k+2, 4k, 2k^2$, $k > 0$. Find the value of k.

(4 marks)

$$\frac{4k}{k+2} = \frac{2k^2}{4k}$$

$$\frac{4k}{k+2} = \frac{k}{2}$$

$$8k = k(k+2)$$

$$k^2 + 2k - 8k = 0$$

$$k^2 - 6k = 0$$

$$k(k-6) = 0$$

~~$k=0$~~ $k = 6$

5. $$\mathrm{f}(x)=\frac{x^4+2x^3-29x^2-47x+77}{x^2-2x-15}$$

Show that $\mathrm{f}(x)$ can be written as $Px^2+Qx+R+\dfrac{V}{x+3}+\dfrac{W}{x-5}$ and find the values of P, Q, R, V and W.

(7 marks)

$$\begin{array}{r}
x^2+4x-6 \\
x^2-2x-15\,\big)\,x^4+2x^3-29x^2-47x+77 \\
-\;x^4-2x^3-15x^2 \\
\hline
4x^3-14x^2-47x+77 \\
-\;4x^3-8x^2-60x \\
\hline
-6x^2+13x+77 \\
-6x^2+12x+90 \\
\hline
x-13
\end{array}$$

$x^2-2x-15$

$$f(x)=x^2+4x-6+\frac{x-13}{x^2-2x-15}$$

$$\frac{x-13}{(x-5)(x+3)}\equiv\frac{A}{x+3}+\frac{B}{x-5}$$

$$x-13\equiv A(x-5)+B(x+3)$$

let $x=5$

$$-8=B(8)$$

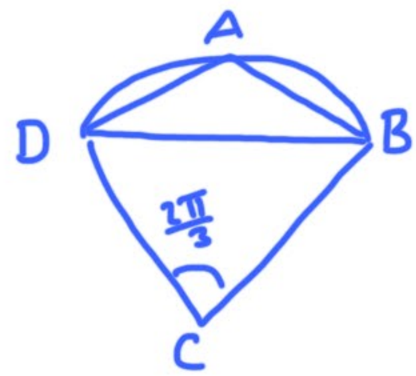
$\underline{B=-1}$

let $x=-3$

$$-16=A(-3-5)\rightarrow \underline{A=2}$$

$$f(x)=x^2+4x-6+\frac{2}{x+3}-\frac{1}{x-5}$$

6. Figure 1 shows a logo comprised of a rhombus surrounded by two arcs. Arc *BAD* has centre *C* and arc *BCD* has centre *A*. Some of the dimensions of the logo are shown in the diagram.



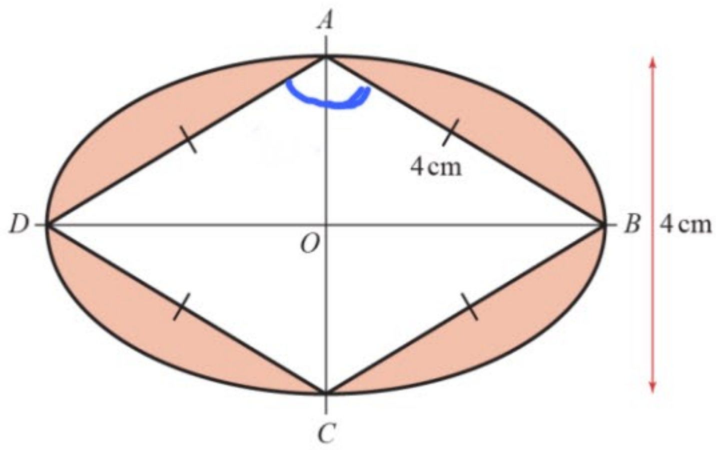


Figure 1

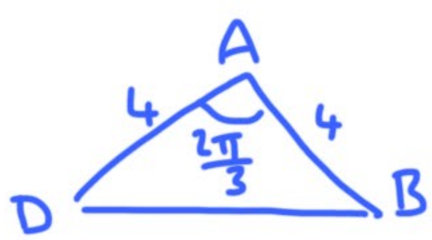
$OAB = \cos^{-1}\left(\frac{2}{4}\right)$

$= \frac{\pi}{3}$

Prove that the shaded area of the logo is $\frac{2}{3}\left(16\pi - 24\sqrt{3}\right)$

(8 marks)

Area of sector CDAB $= \frac{1}{2} \times 4^2 \times \frac{2\pi}{3} = \frac{16\pi}{3}$



Area DAB $= \frac{1}{2} \times 4 \times 4 \times \sin\frac{2\pi}{3} = 4\sqrt{3}$

$\frac{16\pi}{3} - 2\left(4\sqrt{3}\right) = \frac{16\pi}{3} - 8\sqrt{3}$

So Shaded area $= 2\left(\frac{16\pi}{3} - \frac{24}{3}\sqrt{3}\right)$

$= \frac{2}{3}\left(16\pi - 24\sqrt{3}\right)$

7. C has parametric equations $x = \dfrac{1+4t}{1-t}$, $y = \dfrac{2+bt}{1-t}$, $-1 \leqslant t \leqslant 0$.

(a) Show that the cartesian equation of C is $y = \left(\dfrac{2+b}{5}\right)x + \left(\dfrac{8-b}{5}\right)$, over an appropriate domain.

(4 marks)

$$x = \frac{1+4t}{1-t}$$

$$x(1-t) = 1+4t$$

$$x - xt = 1+4t$$

$$x - 1 = xt + 4t$$

$$x - 1 = t(x+4)$$

$$t = \frac{x-1}{x+4}$$

$$y = \frac{2+bt}{1-t}$$

$$y = \frac{2 + b\left(\frac{x-1}{x+4}\right)}{1 - \left(\frac{x-1}{x+4}\right)}$$

$$y = \frac{2(x+4) + b(x-1)}{x+4-(x-1)}$$

$$= \frac{2x+8+bx-b}{x+4-x+1}$$

$$= \frac{2x+bx+8-b}{5}$$

$$y = \frac{(2+b)x}{5} + \frac{8-b}{5}$$

7. C has parametric equations $x = \dfrac{1+4t}{1-t}$, $y = \dfrac{2+bt}{1-t}$, $-1 \leqslant t \leqslant 0$.

(a) Show that the cartesian equation of C is $y = \left(\dfrac{2+b}{5}\right)x + \left(\dfrac{8-b}{5}\right)$, over an appropriate domain.

(4 marks)

Given that C is a line segment and that the gradient of the line is -1,

(b) show that the length of the line segment is $a\sqrt{2}$, where a is a rational number to be found.

(4 marks)

$\dfrac{2+b}{5} = -1$ $\quad C: \; y = -x + 3 \quad (1,2)$

$2+b = -5$ $\quad$ when $t = 0 \quad x = 1 \quad y = 2$

$b = -7$ $\quad$ when $t = -1 \quad x = \dfrac{-3}{2} \quad y = \dfrac{2+7}{2} = \dfrac{9}{2}$

$\left(-\dfrac{3}{2}, \dfrac{9}{2}\right)$

$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$\sqrt{\left(1 - -\dfrac{3}{2}\right)^2 + \left(2 - \dfrac{9}{2}\right)^2} = \sqrt{\dfrac{25}{4} + \dfrac{25}{4}} = \dfrac{\sqrt{50}}{2} = \dfrac{5\sqrt{2}}{2}$

8. A toy soldier is connected to a parachute. The soldier is thrown into the air from ground level. The height, in metres, of the soldier above the ground can be modelled by the equation $H = \frac{4t^{\frac{2}{3}}}{t^2+1}$, $0 \leqslant t \leqslant 6$ s, where H is height of the soldier above the ground and t is the time since the soldier was thrown.

(a) Show that $\frac{dH}{dt} = \frac{8(1-2t^2)}{3\sqrt[3]{t}(t^2+1)^2}$.

$\frac{u'v - uv'}{v^2}$

(4 marks)

$$\frac{dH}{dt} = \frac{\frac{8}{3}t^{-1/3}(t^2+1) - 2t \times 4t^{2/3}}{(t^2+1)^2} \times \frac{3\sqrt[3]{t}}{3\sqrt[3]{t}}$$

$$= \frac{\frac{8}{3\sqrt[3]{t}}\left(t^2+1 - 3 \times t^1 \times t^{2/3} \times t^{1/3}\right)}{(t^2+1)^2}$$

$$= \frac{8(t^2+1-3t^2)}{3\sqrt[3]{t}(t^2+1)^2} = \frac{8(1-2t^2)}{3\sqrt[3]{t}(t^2+1)^2}$$

(b) Using the differentiated function, explain whether the soldier was increasing or decreasing in height after 2 seconds.

(2 marks)

when $t=2$

$$\frac{dH}{dt} = \frac{8(1-8)}{3\sqrt[3]{2}(25)} \approx -0.593$$

negative value so the soldier is decreasing in height.

8. A toy soldier is connected to a parachute. The soldier is thrown into the air from ground level. The height, in metres, of the soldier above the ground can be modelled by the equation $H = \dfrac{4t^{\frac{2}{3}}}{t^2+1}$, $0 \le t \le 6$ s, where H is height of the soldier above the ground and t is the time since the soldier was thrown.

(a) Show that $\dfrac{\mathrm{d}H}{\mathrm{d}t} = \dfrac{8(1-2t^2)}{3\sqrt[3]{t}\,(t^2+1)^2}$.

(4 marks)

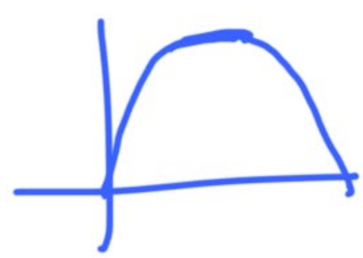
(b) Using the differentiated function, explain whether the soldier was increasing or decreasing in height after 2 seconds.

(2 marks)

(c) Find the exact time when the soldier reaches a maximum height.

(2 marks)

$$\frac{dH}{dt} = \frac{8(1-2t^2)}{3\sqrt[3]{t}(t^2+1)^2} = 0$$



$$8(1-2t^2) = 0$$

$$1 - 2t^2 = 0$$

$$2t^2 = 1$$

$$t^2 = \frac{1}{2}$$

$t = \dfrac{1}{\sqrt{2}}$ seconds or $\dfrac{\sqrt{2}}{2}$ seconds

9. (a) Show that $\tan^4 x \equiv \sec^2 x \tan^2 x + 1 - \sec^2 x$.

(4 marks)

$\tan^2 x \tan^2 x$

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$\tan^2 x (\sec^2 x - 1)$

$= \tan^2 x \sec^2 x - \tan^2 x$

$= \tan^2 x \sec^2 x - (\sec^2 x - 1)$

$= \tan^2 x \sec^2 x - \sec^2 x + 1$

$= \sec^2 x \tan^2 x + 1 - \sec^2 x$

(b) Hence find the exact value of $\int_0^{\frac{\pi}{4}} \tan^4 x \, dx$.

(5 marks)

$\int_0^{\pi/4} (\sec^2 x \tan^2 x + 1 - \sec^2 x) \, dx$

$\int \sec^2 x \tan^2 x \, dx$

let $u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$dx = \frac{du}{\sec^2 x}$

$= \int \sec^2 x \, u^2 \frac{du}{\sec^2 x}$

$= \int u^2 \, du = \frac{u^3}{3} = \frac{\tan^3 x}{3}$

$= \left[\frac{\tan^3 x}{3} + x - \tan x \right]_0^{\pi/4}$

$= \left(\frac{1}{3} + \frac{\pi}{4} - 1 \right) - (0)$

$= \frac{\pi}{4} - \frac{2}{3}$

10. Use proof by contradiction to show that, given a rational number a and an irrational number b, $a - b$ is irrational.

(4 marks)

Given that a is rational, b is irrational, assume that a-b is rational

If a is rational $a = \frac{m}{n}$

$a - b = \frac{p}{q}$

$\frac{m}{n} - b = \frac{p}{q}$

$b = \frac{m}{n} - \frac{p}{q}$

$= \frac{mq}{nq} - \frac{np}{nq}$

$b = \frac{mq - np}{nq}$ ← this is rational

but we assumed b is irrational which is a contradiction.

So a-b must be irrational if b is irrational

11. $f(x) = |2x + 3| - 4, \ x \in \mathbb{R}.$

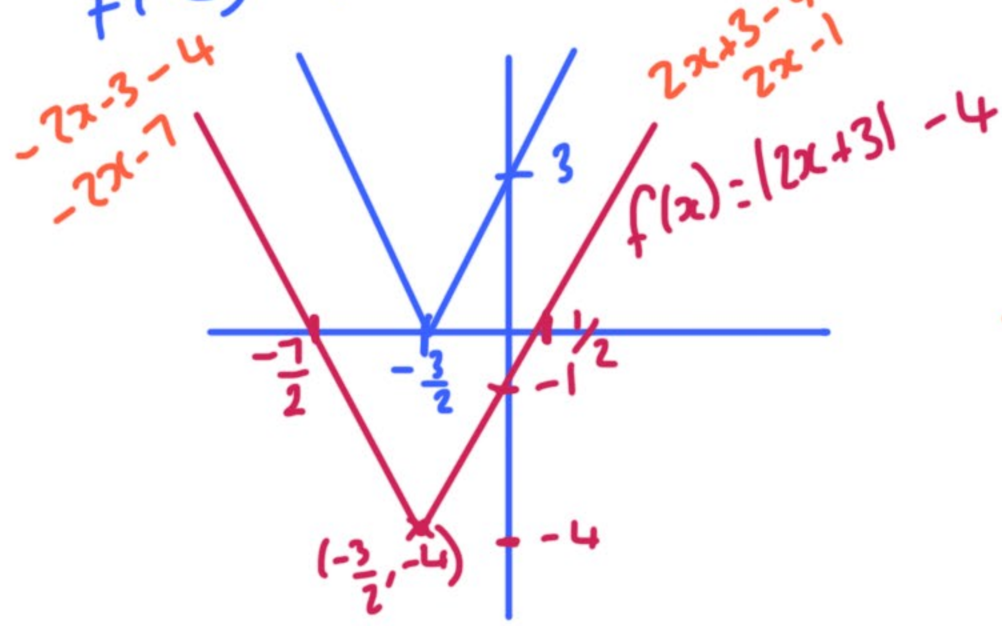
$2x - \frac{7}{2}$

$|-4| = 4$

(a) Sketch the graph of $y = f(x)$, labelling its vertex and any points of intersection with the coordinate axes.

(5 marks)

$f(x) = |2x + 3|$



$2x + 3 = 0$

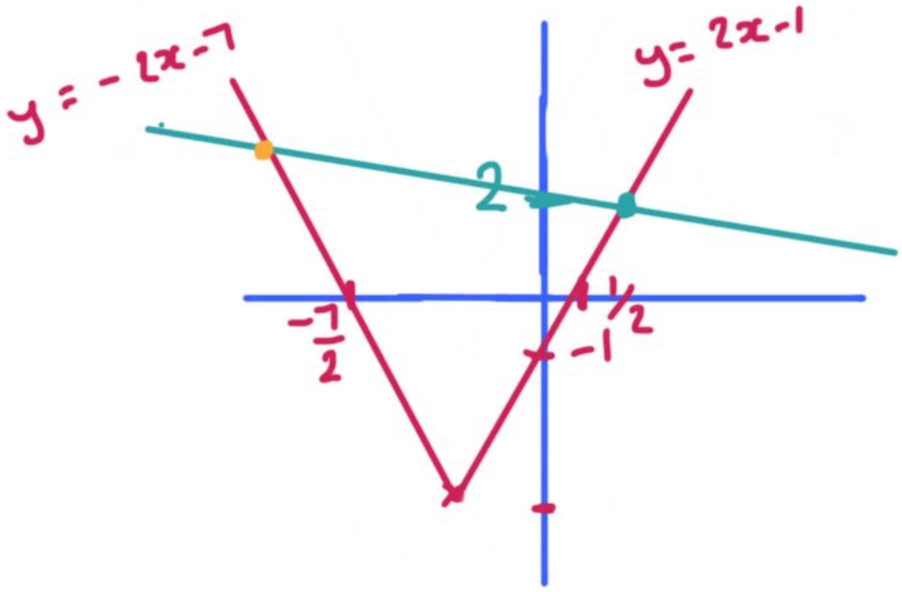
$x = -\frac{3}{2}$

$2x - 1 = 0$

$x = \frac{1}{2}$

(b) Find the coordinates of the points of intersection of $y = |2x+3| - 4$ and $y = -\frac{1}{4}x + 2$.

(5 marks)



$2x - 1 = -\frac{1}{4}x + 2$

$8x - 4 = -x + 8$

$9x = 12$

$x = \frac{12}{9} = \frac{4}{3}$

$y = \frac{8}{3} - 1 = \frac{5}{3}$

$\left(\frac{4}{3}, \frac{5}{3}\right)$

$-2x - 7 = -\frac{1}{4}x + 2$

$-8x - 28 = -x + 8$

$-7x = 36$

$x = -\frac{36}{7}$

$y = \frac{72}{7} - 7$

$y = \frac{23}{7}$

$\left(-\frac{36}{7}, \frac{23}{7}\right)$

12. (a) Prove that $(\sin 3\theta + \cos 3\theta)^2 = 1 + \sin 6\theta$

(3 marks)

$(\sin 3\theta + \cos 3\theta)(\sin 3\theta + \cos 3\theta)$

$= \sin^2 3\theta + 2\sin 3\theta \cos 3\theta + \cos^2 3\theta$

$= 1 + 2\sin 3\theta \cos 3\theta$

$= 1 + \sin 6\theta$

$\sin 2x = 2\sin x \cos x$

$\sin 6\theta = 2\sin 3\theta \cos 3\theta$

(b) Use the result to solve, for $0 \le \theta \le \frac{\pi}{2}$, the equation $(\sin 3\theta + \cos 3\theta) = \sqrt{\frac{2+\sqrt{2}}{2}}$.

Give your answer in terms of π. Check for extraneous solutions.

(4 marks)

$\sqrt{1 + \sin 6\theta} = \sqrt{\frac{2+\sqrt{2}}{2}}$

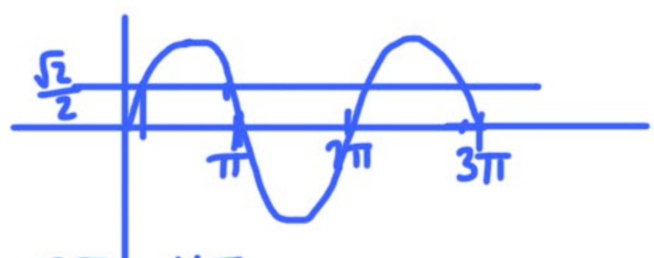
~~= 1.307~~

$1 + \sin 6\theta = \frac{2+\sqrt{2}}{2}$

$\sin 6\theta = \frac{2+\sqrt{2}}{2} - \frac{2}{2}$

$= \frac{2+\sqrt{2}-2}{2}$

$\sin 6\theta = \frac{\sqrt{2}}{2}$



$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = 6\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$

$\theta = \frac{\pi}{24}, \frac{3\pi}{24}, \frac{9\pi}{24}, \frac{11\pi}{24}$

$\theta = \frac{\pi}{24}, \frac{\pi}{8}$, ~~$\frac{3\pi}{8}$~~, ~~$\frac{11\pi}{24}$~~

$\sin 3\theta + \cos 3\theta = \sqrt{\frac{2+\sqrt{2}}{2}}$ $(= 1.307)$

13. $$f(x) = \frac{6}{2+3x} - \frac{4}{3-5x}, \quad |x| < \frac{3}{5}.$$

(a) Show that the first three terms in the series expansion of f(x) can be written as $\frac{5}{3} - \frac{121}{18}x + \frac{329}{108}x^2$.

(7 marks)

$6(2+3x)^{-1}$

$6\left(2\left(1+\frac{3x}{2}\right)\right)^{-1} = 6\left(2^{-1}\left(1+\frac{3x}{2}\right)^{-1}\right)$

$3\left(1+\frac{3x}{2}\right)^{-1} = 3\left(1+(-1)\left(\frac{3x}{2}\right) + \frac{(-1)(-2)\left(\frac{3x}{2}\right)^2}{2}\right)$

$= 3\left(1 - \frac{3x}{2} + \frac{9x^2}{4}\right) = 3 - \frac{9x}{2} + \frac{27x^2}{4}$

$4(3-5x)^{-1} = 4 \times 3^{-1}\left(1-\frac{5x}{3}\right)^{-1}$

$\frac{4}{3}\left(1-\frac{5x}{3}\right)^{-1} = \frac{4}{3}\left(1+(-1)\left(-\frac{5x}{3}\right) + \frac{(-1)(-2)\left(-\frac{5x}{3}\right)^2}{2}\right)$

$= \frac{4}{3}\left(1 + \frac{5x}{3} + \frac{25x^2}{9}\right) = \frac{4}{3} + \frac{20x}{9} + \frac{100x^2}{27}$

$3 - \frac{9x}{2} + \frac{27x^2}{4}$

$-\frac{4}{3} - \frac{20x}{9} - \frac{100x^2}{27}$

$\frac{5}{3} - \frac{121x}{18} + \frac{329x^2}{108}$

(b) Find the exact value of f(0.01). Round your answer to 7 decimal places.

(2 marks)

$f(0.01) = \frac{6}{2+0.03} - \frac{4}{3-0.05} = 1.5997328$ (7dp)

(c) Find the percentage error made in using the series expansion in part (a) to estimate the value of f(0.01).

Give your answer to 2 significant figures.

(3 marks)

$\frac{5}{3} - \frac{121}{8} \times 0.01 + \frac{329}{108} \times 0.01^2 = 1.5997491$

$\frac{1.5997491 - 1.5997328}{1.5997328} \times 100 = 0.0010 \text{ (2sf)}$

14. Jacob is making some patterns out of squares. The first 3 patterns in the sequence are shown in Figure 2.

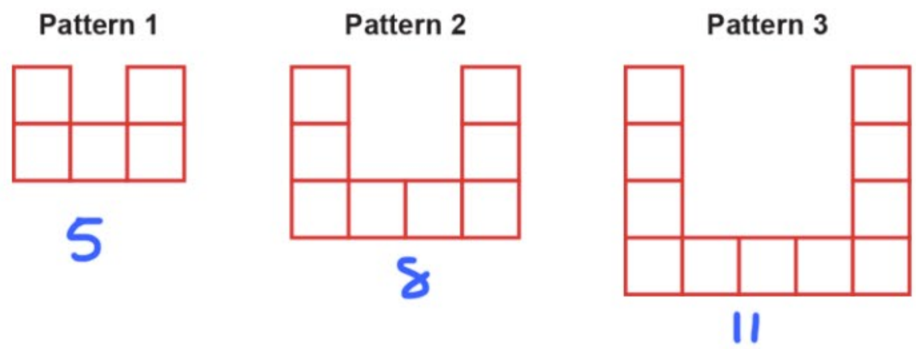


Figure 2

(a) Find an expression, in terms of n, for the number of squares required to make pattern n.

(2 marks)

5, 8, 11

$3n + 2$

(b) Show that $3k^2 + 7k - 1896 = 0$.

(2 marks)

$S_k = 948$

$a = 5$

$d = 3$

5, 8, 11, ...

$S_k = \frac{k}{2}(10 + (k-1)3) = 948$

$k(10 + 3k - 3) = 1896$

$k(7 + 3k) = 1896$

$7k + 3k^2 = 1896$

$3k^2 + 7k - 1896 = 0$

15. Figure 3 shows part of the curve with equation $y = x\sin^2 x$. The finite region bounded by the line with equation $x = \frac{\pi}{2}$, the curve and the x-axis is shown shaded in the diagram.

Find the area of the shaded region.

(7 marks)

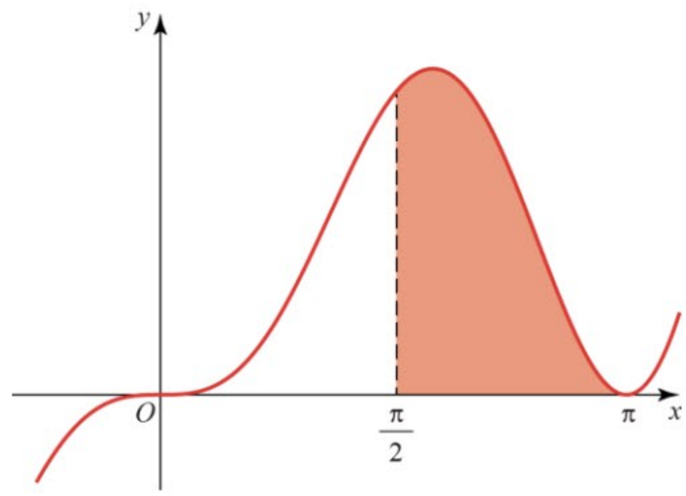


Figure 3

$$\int_{\frac{\pi}{2}}^{\pi} x \sin^2 x \, dx$$

(x ↑ u, $\sin^2 x$ ↑ v')

$$\int uv' = uv - \int u'v$$

$u = x$

$u' = 1$

$v' = \sin^2 x = \frac{1}{2} - \frac{\cos 2x}{2}$

$v = \frac{x}{2} - \frac{\sin 2x}{4}$

$\cos 2x = \cos^2 x - \sin^2 x$

$\cos 2x = 1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos 2x$

$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

$$x\left(\frac{x}{2} - \frac{\sin 2x}{4}\right) - \int \left(\frac{x}{2} - \frac{\sin 2x}{4}\right) dx$$

$+ \int \frac{\sin 2x}{4}$

$$= \left[\frac{x^2}{2} - \frac{x\sin 2x}{4} - \frac{x^2}{4} - \frac{\cos 2x}{8}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \left(\frac{\pi^2}{2} - 0 - \frac{\pi^2}{4} - \frac{1}{8}\right) - \left(\frac{\pi^2}{8} - 0 - \frac{\pi^2}{16} + \frac{1}{8}\right)$$

$$= \pi^2\left(\frac{1}{2} - \frac{1}{4} - \frac{1}{8} + \frac{1}{16}\right) - \frac{1}{8} - \frac{1}{8}$$

$$= \frac{3\pi^2}{16} - \frac{1}{4}$$