A2 Practice Paper B.

1. Proof by contradiction:
Opposite is true
Assumption: there exists a product of two odd numbers that is even.

$2m+1 \quad 2n+1$

$(2m+1)(2n+1) = 4nm + 2m + 2n + 1$

$= 2(2nm + m + n) + 1$

$\therefore$ odd

Contradicts the assumption that the product of two odd numbers is even, therefore the product of two odd numbers is odd.

2a) $S = a + (a+d) + (a+2d) + \dots + (a+(n-1)d)$

$S = (a+(n-1)d) + (a+(n-1)d) + (a+(n-3)d) + \dots + a$

Ascending + descending

$2S = (2a + (n-1)d)n$

$S = \frac{n}{2}(2a + (n-1)d)$.

b) $\left.\begin{array}{l} n = 200 \\ a = 1 \\ d = 2 \end{array}\right\}$ $S = \frac{200}{2}(2 + 199(2))$

$S = 40,000$

3. $y = \ln 3x - e^{-2x}$

$$\boxed{\begin{aligned} y &= \ln f(x) \\ \frac{dy}{dx} &= \frac{f'(x)}{f(x)} \end{aligned}}$$

$$\frac{dy}{dx} = \frac{3}{3x} + 2e^{-2x}$$

$$\frac{dy}{dx} = \frac{1}{x} + 2e^{-2x}$$

$$x = 1, \quad \frac{dy}{dx} = \frac{1}{1} + 2e^{-2}$$
$$= 1 + \frac{2}{e^2}$$
$$= \frac{e^2}{e^2} + \frac{2}{e^2} = \frac{2+e^2}{e^2}$$

$$x = 1, \quad y = \ln 3 - e^{-2}$$
$$= \ln 3 - \frac{1}{e^2}$$

$$y - y_1 = m(x - x_1)$$
$$y - \left(\ln 3 - \frac{1}{e^2}\right) = \frac{e^2+2}{e^2}(x-1)$$
$$y - \left(\ln 3 - \frac{1}{e^2}\right) = \left(\frac{e^2+2}{e^2}\right)x - \frac{e^2+2}{e^2}$$
$$y = \left(\frac{e^2+2}{e^2}\right)x - \left(\frac{e^2+2}{e^2}\right) + \ln 3 + \frac{1}{e^2}$$
$$y = \left(\frac{e^2+2}{e^2}\right)x - \left(\frac{e^2+2}{e^2} + \frac{1}{e^2}\right) + \ln 3.$$
$$y = \left(\frac{e^2+2}{e^2}\right)x - \left(\frac{e^2+3}{e^2}\right) + \ln 3$$

4. $x = 7\sin t - 4 \qquad y = 7\cos t + 3 \qquad -\frac{\pi}{2} < t < \frac{\pi}{3}$

a) $\frac{x+4}{7} = \sin t \qquad \frac{y-3}{7} = \cos t$

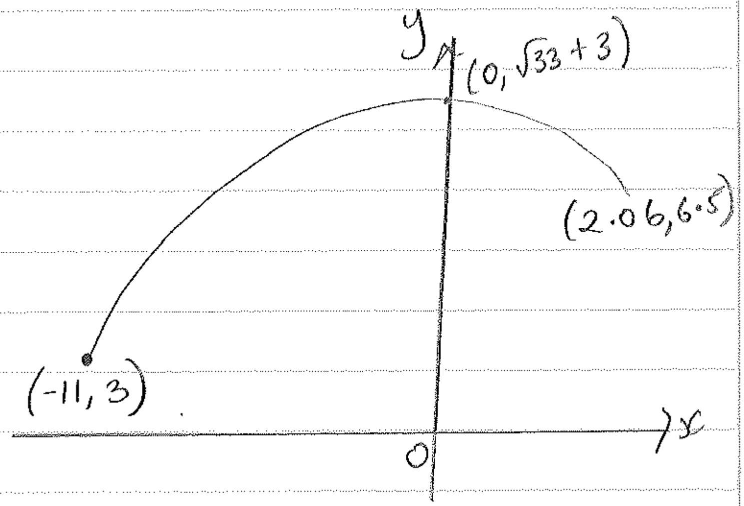
$\sin^2 t + \cos^2 t = 1$

$\left(\frac{x+4}{7}\right)^2 + \left(\frac{y-3}{7}\right)^2 = 1$

$(x+4)^2 + (y-3)^2 = 7^2$

b) Centre of circle $(-4, 3)$

Radius $= 7$



$t = \frac{-\pi}{2} \qquad x = -11$

$y = 3$

$t = \frac{\pi}{3} \qquad x \approx 2.06$

$y = 6.5$

$t = 0 \qquad x = -4$

$y = 10$

$x = 0$

$y = \sqrt{33} + 3$

c) $C = 2\pi r$

$C = 2 \times \pi \times 7 = 14\pi \qquad \frac{3\pi}{6} \quad \frac{2\pi}{6}$

Arc is $\frac{5}{12}$ of circumference

$\frac{5}{12} \times 14\pi = \frac{35}{6}\pi$

5. $A(-1, 7, k)$
$B(4, 1, 10)$

$|\vec{AB}| = 5\sqrt{5}$

a) $\sqrt{5^2 + (-6)^2 + (10-k)^2} = 5\sqrt{5}$
$\sqrt{25 + 36 + (10-k)(10-k)} = 5\sqrt{5}$
$61 + 100 + k^2 - 20k = 25 \times 5$
$k^2 - 20k + 161 = 125$
$k^2 - 20k + 36 = 0$
$(k-2)(k-18) = 0$
$k = 2 \quad k = 18$

b) $A(-1, 7, 18)$

$|\vec{OA}| = \sqrt{(-1)^2 + 7^2 + 18^2}$
$= \sqrt{374}$

Unit vector $\frac{1}{\sqrt{374}}(-i + 7j + 18k)$

6. $\int_{\ln 2}^{\ln b} \left(\frac{e^{2x}}{e^{2x}-1} \right) dx = \ln 4$

$\int \frac{e^{2x}}{e^{2x}-1} = \frac{1}{2} \ln(e^{2x}-1) + c$

$$\boxed{\begin{aligned} \int \frac{f'(x)}{f(x)} &= \ln[f(x)] \\ \int \frac{2e^{2x}}{e^{2x}-1} &= \ln(e^{2x}-1) \\ \int \frac{e^{2x}}{e^{2x}-1} &= \frac{1}{2} \ln(e^{2x}-1) \end{aligned}}$$

$\left[\frac{1}{2} \ln(e^{2x}-1) \right]_{\ln 2}^{\ln b}$

$\left(\frac{1}{2} \ln(e^{2\ln b}-1) \right) - \left(\frac{1}{2} \ln(e^{2\ln 2}-1) \right)$

$\left(\frac{1}{2} \ln(e^{\ln b^2}-1) \right) - \left(\frac{1}{2} \ln(e^{\ln 2^2}-1) \right) = \ln 4.$

$\frac{1}{2} \ln(b^2-1) - \frac{1}{2} \ln(2^2-1) = \ln 4$

$\frac{1}{2} \ln(b^2-1) - \frac{1}{2} \ln 3 = \ln 4$

$\ln(b^2-1) - \ln 3 = 2\ln 4$

$\ln\left(\frac{b^2-1}{3} \right) = \ln 4^2$

$\frac{b^2-1}{3} = 16$

$b^2 - 1 = 48$

$b^2 = 49$

$\underline{b = 7}$

7.a) $x_1 = 4 \quad x_{n+1} = px_n - 9$

$x_2 = 4p - 9$
$x_3 = p(4p-9) - 9$
$= 4p^2 - 9p - 9$

b) $4p^2 - 9p - 9 = 46$
$4p^2 - 9p - 55 = 0.$
$(4p + 11)(p - 5) = 0$
$4p + 11 = 0$
$4p = -11$
$p \neq \frac{-11}{4} \quad \underline{\underline{p = 5}}$

p is an integer.

c) $x_3 = 46 \qquad x_{n+1} = 5x_n - 9.$
$x_4 = 5(46) - 9 = 221$
$x_5 = 5(221 - 9 = \underline{1096}$

8. $\frac{6}{4x^2 + 8x - 5} + \frac{3x + 1}{2x - 1}$

$\frac{6}{(2x-1)(2x+5)} + \frac{3x+1}{2x-1} \frac{\times(2x+5)}{\times(2x+5)}$

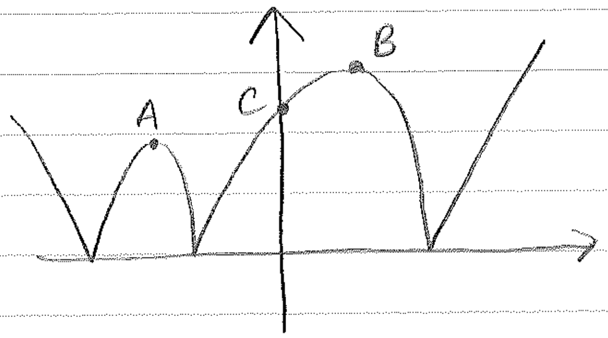
$\frac{6 + (3x+1)(2x+5)}{(2x-1)(2x+5)}$

$\frac{6 + 6x^2 + 2x + 15x + 5}{(2x-1)(2x+5)}$

$\frac{6x^2 + 17x + 11}{(2x-1)(2x+5)} = \frac{(6x+11)(x+1)}{(2x-1)(2x+5)}.$

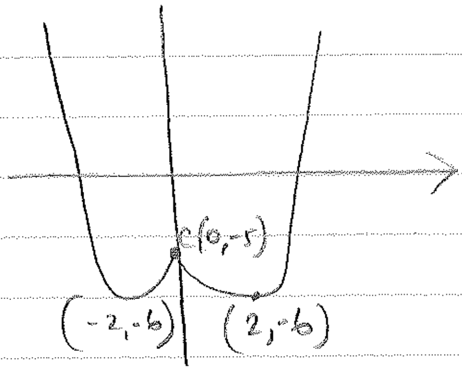
9 a) $y = |f(x)|$

A $(-4, 3) \rightarrow (-4, 3)$
B $(2, -6) \rightarrow (2, 6)$
C $(0, -5) \rightarrow (0, 5)$



b) $y = f(|x|)$

A $(-4, 3) \rightarrow$ Not possible.
B $(2, -6) \rightarrow (2, -6)$
C $(0, -5) \rightarrow (0, -5)$



c) $y = 2f(x+3)$

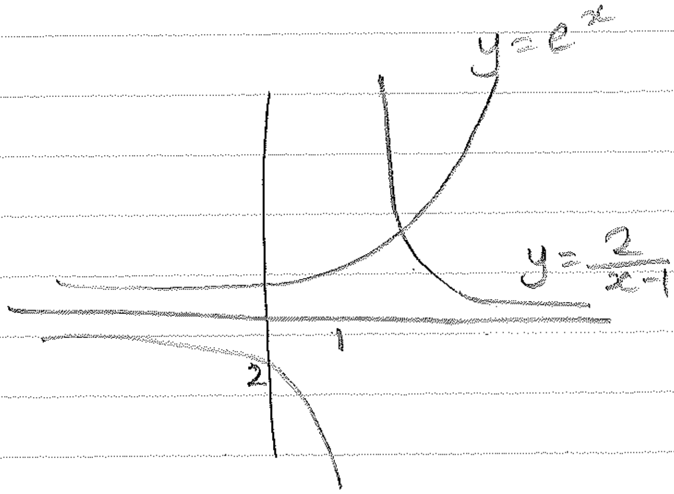
A $(-4, 3) \rightarrow (-7, 6)$ (incorrect on mark scheme).
B $(2, -6) \rightarrow (-1, -12)$
C $(0, -5) \rightarrow (-3, -10)$

10 a) $g(x) = \frac{2}{x-1} - e^x$

$0 = \frac{2}{x-1} - e^x$

$e^x = \frac{2}{x-1}$

$y = e^x \qquad y = \frac{2}{x-1}$



Graphs cross at one point, therefore $\frac{2}{x-1} = e^x$ has one root.

10b) $\frac{2}{x-1} - e^{x} = 0$

$\frac{2}{x-1} = e^{x}$

$2 = e^{x}(x-1)$

$\frac{2}{e^{x}} = x-1$

$\frac{2}{e^{x}} + 1 = x$

$2e^{-x} + 1 = x.$

c) $x_0 = 1.5$

$x_{n+1} = 2e^{-x_n} + 1$

$x_1 = 1.4463$
$x_2 = 1.4709$
$x_3 = 1.4594$
$x_4 = 1.4647.$

d) Newton - Raphson: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$g(x) = \frac{2}{x-1} - e^{x}$

$g(x) = 2(x-1)^{-1} - e^{x}$

$g'(x) = -2(x-1)^{-2} - e^{x}$
$= \frac{-2}{(x-1)^{2}} - e^{x}$

$g(1.5) = -0.4816.$
$g'(1.5) = -12.4816$

10d) continued...

$$x_1 = 1.5 - \frac{-0.4816}{-12.4816}$$

$$\underline{x_1 = 1.461}$$

11.a) $\dfrac{1+x}{\sqrt{1-2x}} = (1+x)(1-2x)^{-1/2}$

$$(1-2x)^{-\frac{1}{2}} = 1 + \left(-\tfrac{1}{2}\right)(-2x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\times 2}(-2x)^2$$

$$= 1 + x + \frac{3x^2}{2}$$

$$(1+x)\left(1 + x + \frac{3}{2}x^2\right)$$

$$= 1 + x + \frac{3}{2}x^2$$

$$\underline{\quad + x + x^2}$$

$$\underline{= 1 + 2x + \frac{5}{2}x^2}$$

b) $|2x| < 1$

$|x| < \frac{1}{2}$

c) $x = \dfrac{1}{100}$, $\dfrac{1 + \frac{1}{100}}{\sqrt{1 - \frac{2}{100}}} = \dfrac{\frac{101}{100}}{\sqrt{\frac{98}{100}}} = \dfrac{101\sqrt{2}}{140}$

d) $x = \dfrac{1}{100}$, $1 + \dfrac{2}{100} + \dfrac{5}{2}\left(\dfrac{1}{100}\right)^2 \simeq 1.02025$

$$\frac{101\sqrt{2}}{140} = 1.02025$$

$$\sqrt{2} = (1.02025 \times 140) \div 101$$

$$\sqrt{2} \simeq 1.41421$$

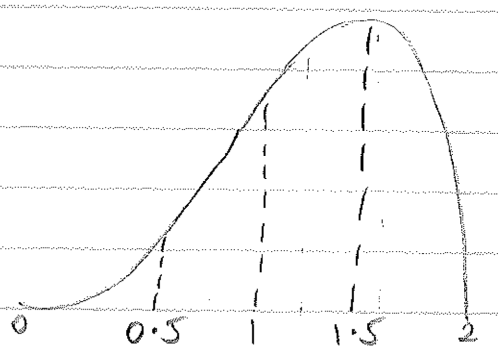
12 a) $y = \frac{1}{2}x^3\sqrt{4-x^2}$

$x = 1.5, \quad y = \frac{1}{2}(1.5)^3\sqrt{4-(1.5)^2}$

$y = 2.23235..$

b) Trap rule :

$A = \frac{h}{2}\left(y_0 + 2(y_1 + y_2 + y_3) + y_4\right)$



h = Strip width = 0.5

$A = \frac{0.5}{2}\left(0 + 2(0.12103 + 0.86603 + 2.23235) + 0\right)$

$\underline{A = 1.610}$ (4sf)

12c) $\int_0^2 \left(\frac{1}{2} x^3 \sqrt{4-x^2}\right) dx$

$u = 4 - x^2$ $\quad\quad$ $x = 0 \rightarrow u = 4$

$\frac{du}{dx} = -2x$ $\quad\quad$ $x = 2 \rightarrow u = 0.$

$\frac{du}{-2x} = dx.$

$\int_4^0 \frac{1}{2} x^3 \sqrt{u} \frac{du}{-2x}.$

$\frac{-1}{4} \int_4^0 x^3 \sqrt{u} \frac{du}{x}.$

$\frac{-1}{4} \int_4^0 x^2 \sqrt{u}\, du$ $\quad\quad$ $u = 4 - x^2$
$x^2 = 4 - u$

$-\frac{1}{4} \int_4^0 (4-u) u^{\frac{1}{2}} du$

$-\frac{1}{4} \int_4^0 4u^{\frac{1}{2}} - u^{3/2} du$

$-\frac{1}{4} \left[\frac{8u^{3/2}}{3} - \frac{2u^{5/2}}{5}\right]_4^0$

$-\frac{1}{4} \left[\left(\frac{8}{3}(0)^{3/2} - \frac{2}{5}(0)^{5/2}\right) - \left(\frac{8}{3}(4)^{3/2} - \frac{2}{5}(4)^{5/2}\right)\right]$

$= \frac{32}{15}.$

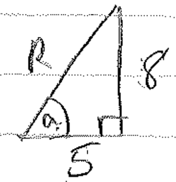
1d) Use more strips to improve the accuracy.

13a) $5\cos\theta - 8\sin\theta \rightarrow R\cos(\theta + a)$

$5\cos\theta - 8\sin\theta = R\cos\theta\cos a - R\sin\theta\sin a$

$5 = R\cos a \qquad 8 = R\sin a.$

$\frac{5}{R} = \cos a \qquad \frac{8}{R} = \sin a.$



$\tan a = \frac{8}{5}$

$a = 1.0122$

$5^2 + 8^2 = R^2$

$25 + 64 = R^2$

$89 = R^2$

$\sqrt{89} = R$

$\sqrt{89}\cos(\theta + 1.0122)$

b) $T = 1100 + 5\cos\left(\frac{x}{3}\right) - 8\sin\left(\frac{x}{3}\right)$

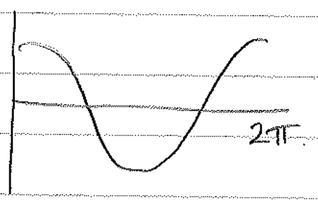
$T = 1100 + \sqrt{89}\cos\left(\frac{x}{3} + 1.0122\right)$

Max at $\cos x = 1$

$T = 1100 + \sqrt{89} \times 1$

$T = 1109.43.$

$\cos\left(\frac{x}{3} + 1.0122\right) = 1.$



$\frac{x}{3} + 1.0122 = 2\pi$

$x = 15.81$ hrs.

$\frac{x}{3} + 1.1022 \neq 0$

As $x > 0.$

12c) $1097 = 1100 + \sqrt{89}\cos\left(\frac{x}{3} + 1.0122\right)$

$-3 = \sqrt{89}\cos\left(\frac{x}{3} + 1.0122\right)$

$\frac{-3}{\sqrt{89}} = \cos\left(\frac{x}{3} + 1.0122\right)$

$\frac{x}{3} + 1.0122 = 1.8944,\ 2\pi - 1.8944,\ 2\pi + 1.8944$

$x = 2.65$ hrs, 10.13 hrs, 21.50 hrs.